Tools for Symmetric Key Provable Security

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Outline of the talk

1. Probability in Cryptography
   - Well Known Distribution in Cryptography
   - Some Metrics on Probability Distributions

2. Two Tools: H-Coefficient and $\chi^2$
   - H-Coefficient Technique
   - Mirror theory
   - $\chi^2$ Method

3. Some Constructions and Applications
   - Encrypted Davies-Meyer (EDM) Construction
   - Truncation Construction
   - Sum of Permutations Construction
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Notations for Probability

1. $X \leftarrow \Omega$: $X$ is a random variable with sample space $\Omega$.

2. $\Pr_X$ denotes the probability function of $X$.

3. For an event $E \subseteq \Omega$ we denote the probability of the event $E$ realized by $X$ as

\[
\Pr_X(E \mid F) = \frac{\Pr_X(E \cap F)}{\Pr_X(F)}.
\]

4. $\Pr_X(E \mid F)$ is the conditional probability defined only when $\Pr_X(F)$ is positive and it is defined as

\[
\Pr_X(E \mid F) = \Pr_X(E \cap F) / \Pr_X(F).
\]
Notations for Probability

1. \( x^t := (x_1, \ldots, x_t) \) for any positive \( t \).
   \( X^t := (X_1, \ldots, X_t) \leftarrow \Omega = \Omega_1 \times \cdots \times \Omega_t \) is also called joint random variable.

2. We denote \( \Pr(X_i = x_i \mid X^{i-1} = x^{i-1}) \) as \( \Pr_{X}(x_i \mid x^{i-1}) \).

3. Let \( X \leftarrow \Omega, f : \Omega \rightarrow \mathbb{R} \) then
   \[
   \mathbb{E}(f(X)) = \sum_{x \in \Omega} f(x) \Pr_{X}(x).
   \]

4. If \( X \) is a real valued random variable
   \[
   
   \text{Var}(X) = E((X - \mathbb{E}(X))^2).
   \]
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Examples. In statistics with replacement (WR) and without replacement sample (WOR) sampling are very popular.

1. \( U := (U_1, \ldots, U_t) \leftarrow \text{wr} \ S \) says that \( U \leftarrow S^t \). So we specify \( \Pr_U \) completely as \( \Pr_U(x^t) = |S|^{-t} \).

2. WOR sample \( V := (V_1, \ldots, V_t) \leftarrow \text{wor} \ S \) is specified through conditional probability as

\[
\Pr_V(x_i \mid x_{i-1}) = \frac{1}{|S|^{-i+1}}, \text{ for all distinct } x_1, \ldots, x_i \in S.
\]
With and Without Replacement Sample

1. **Examples.** In statistics with replacement (WR) and without replacement sample (WOR) sampling are very popular.

2. $U := (U_1, \ldots, U_t) \leftarrow_{\text{wr}} S$ says that $U \leftarrow_S S^t$. So we specify $\Pr_U$ completely as $\Pr_U(x^t) = |S|^{-t}$.

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$$
Why do we study WR and WOR in Cryptography?

1. Let $f \leftarrow \$ \text{Func}(D, R)$ (random function). Then, for any distinct $x_1, \ldots, x_q \in D$,

   $$(f(x_1), \ldots, f(x_q)) \leftarrow \text{wr} \ R.$$  

2. If $\pi \leftarrow \$ \text{Perm}(R)$ (random permutation - we use it for block cipher or permutation in the ideal model) then

   $$(\pi(x_1), \ldots, \pi(x_q)) \leftarrow \text{wor} \ R.$$  

3. The both results are true even if $x_i$’s are some functions of $y^{i-1}$ where $y_j = f(x_j)$ (or $y_j = \pi(x_j)$). This happens for adaptive adversary interacting with $f$ or $\pi$. 
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In cryptography blockcipher modeled to be pseudorandom permutation.

This means (using hybrid argument) that we can replace random permutation instead of a blockcipher.

Consider the XOR construction: \( E_K(x\|0) \oplus E_K(x\|1) \).

If we replace blockcipher by random permutation, the output distribution of the XOR construction is same as \( X^t \) where

\[
X_1 = V_1 \oplus V_2, \ldots, X_t = V_{2t-1} \oplus V_{2t}
\]

and

\[
(V_1, \ldots, V_t) \leftarrow \text{wor} \{0, 1\}^n.
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Why do we study WR and WOR in Cryptography?

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$$(V_1, \ldots, V_t) \leftarrow \text{wor} \{0, 1\}^n.$$
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Total variation

Definition

Total variation (or statistical distance) is a metric on the set of probability functions over $\Omega$.

$$\|P_0 - P_1\| = \frac{1}{2} \sum_{x \in \Omega} |P_0(x) - P_1(x)|.$$
Geometric interpretation of Total variation

Total variation between $X$ and $Y = \text{area } A + \text{area } C$.
(Picture courtesy Shoup's book “A Computational Introduction to Number Theory and Algebra”).
Indistinguishability Game and total variation

- $\mathcal{A}$ is a distinguisher - two oracles $\mathcal{O}_1$ and $\mathcal{O}_2$.

- The *advantage* of the adversary in this game, denoted $\text{Adv}_{\mathcal{A}}(\mathcal{O}_1, \mathcal{O}_2)$, is given by

$$\text{Adv}^{\text{dist}}_{\mathcal{O}_1, \mathcal{O}_2}(\mathcal{A}) := |\Pr(\mathcal{A}^{\mathcal{O}_1} \rightarrow 1) - \Pr(\mathcal{A}^{\mathcal{O}_2} \rightarrow 1)|,$$

- If $X^q$ and $Y^q$ denote the outputs of $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively. Then,

$$\text{Adv}^{\text{dist}}_{\mathcal{O}_1, \mathcal{O}_2}(\mathcal{A}) \leq \| \Pr_{X^q} - \Pr_{Y^q} \|.$$
Properties of Total variation

1. $\|P_0 - P_1\| \leq 1$. When equality holds?

2. **Triangle inequality.** Let $P_i$ be the probability function of $X_i$, $i \in [d] \triangleq \{1, 2, \ldots, d\}$ then

$$\|P_1 - P_d\| \leq \|P_1 - P_2\| + \cdots + \|P_{d-1} - P_d\|.$$
Some Examples of Total Variation

We sometimes denote \( d_{TV}(X, Y) = \| \Pr_X - \Pr_Y \| \).

1. Let \( \mathcal{T} \subseteq \mathcal{S} \) and \( X \leftarrow \mathcal{S}, Y \leftarrow \mathcal{T} \). Then,

\[
d_{TV}(X, Y) = 1 - \frac{|\mathcal{T}|}{|\mathcal{S}|}.
\]

2. Let \( |\mathcal{S}| = N \), \( U^q \leftarrow_w r \mathcal{S} \) and \( V^q \leftarrow_w o r \mathcal{S} \) then

\[
d_{TV}(U, V) = 1 - \prod_{i=1}^{q-1} \left( 1 - \frac{i}{N} \right) = cp(q, N)
\]

where \( cp(q, N) \) denotes the collision probability of \( q \) random elements chosen from a set of size \( N \).
Chi-square distance

The $\chi^2$ distance between $P_0$ and $P_1$, with $P_0 \ll P_1$ (support of $P_0$ is contained in that of $P_1$), is defined as

$$d_{\chi^2}(P_0, P_1) := \sum_{x \in \Omega} \frac{(P_0(x) - P_1(x))^2}{P_1(x)}.$$

- Has its origin in mathematical statistics dating back to Pearson.
- It can be seen that $\chi^2$ distance is not symmetric, does not satisfy triangle inequality.
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- Has its origin in mathematical statistics dating back to Pearson.
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Other Metrics

1. Helinger distance: Steinberger used this metric to bound advantage of key-alternating cipher.

2. Renyi divergence of order $a$ (generalized form of $\chi^2$. When $a = 2$ it is closely related to $\chi^2$). Used in lattice based cryptography.

3. Separation measurement (used in Markov chain).

4. KL divergence is popular in cryptography. Also used in the proof of the $\chi^2$ method.
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1. $\mathcal{O}_1$ or $\mathcal{O}_2$ two oracles returning $\mathcal{Y}$ elements.

2. Transcript: $y^q \in \mathcal{Y}^q$.

3. Let $X^q$ and $Y^q$ be the responses while $A$ interacts with $\mathcal{O}_1$ and $\mathcal{O}_2$ respectively.
Theorem (H-coefficient technique)

Let $Y^q = \mathcal{V}_{\text{good}} \sqcup \mathcal{V}_{\text{bad}}$ be a partition. Suppose for any $x^q \in \mathcal{V}_{\text{good}},$

\[
\frac{\Pr(X^q = x^q)}{\Pr(Y^q = x^q)} := \frac{ip_{\text{real}}}{ip_{\text{ideal}}} \geq 1 - \epsilon_{\text{ratio}},
\]

and

\[
\Pr[Y^q \in \mathcal{V}_{\text{bad}}] \leq \epsilon_{\text{bad}}.
\]

Then,

\[
\text{Adv}_{\mathcal{O}_1,\mathcal{O}_2}^\text{dist}(\mathcal{A}) \leq \epsilon_{\text{ratio}} + \epsilon_{\text{bad}}.
\]
Simple Applications

1. PRP-PRF switching lemma.
2. Hash-then-PRF.
3. Hash-then-TBC.
4. Many more...
Summing up H-Coefficient

1. Good tool for birthday bound.

2. Some times we have beyond birthday bound, mostly $2^{3n/4}$ and $2^{2n/3}$ (in case of xor of $k$ permutations we have bound of the form $2^{(2k-1)n/2k}$).

3. Not so powerful for optimal security (i.e., $n$ bit security).

4. Mirror theory for sum of permutation. Not easy to understand the proof. Seems to have non-trivial gaps.
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What is Mirror theory?

1. A combinatorial result.

2. Hall’s result: Let $\mathcal{G}$ be an abelian group and $f : \mathcal{G} \to \mathcal{G}$ be a function such that $\sum_{x \in \mathcal{G}} f(x) = 0$. Then there exists two permutations $\pi_1, \pi_2$ over $\mathcal{G}$ such that $f = \pi_1 - \pi_2$.

3. It has been proved by induction by Marshall J. Hall in 1951.
What is Mirror theory?

1. Patarin extend this with a cryptographic motivation.
2. Number of functions is $N^N$ and the number of permutations is $N!$ where $N = |\mathcal{G}|$.
3. The number of pairs of permutations $(\pi_1, \pi_2)$ such that $f = \pi_1 - \pi_2$ is about $\frac{N!^2}{N^N}$ (on the average).
4. Instead of matching a function exactly, match over a domain of size $q$ (the query set for an adversary).
Patarin claimed for $q < N/67$ and for any $q$-distinct $x^q$, and any (not necessarily distinct) $y_1, \ldots, y_q$ (so no bad transcripts and hence $\epsilon_{\text{bad}} = 0$),

$$\#\{(\pi_1, \pi_2) : \pi_1(x_i) + \pi_2(x_i) = y_i\} \geq \frac{N!^2}{N^q} \times (1 - \epsilon_{\text{ratio}})$$

where $\epsilon_{\text{ratio}} = O(q/2^n)$

In other words,

$$\Pr(\text{RP}_1(x_1) + \text{RP}_2(x_1) = y_1, \ldots, \text{RP}_1(x_q) + \text{RP}_2(x_q) = y_q) \geq \frac{1 - \epsilon_{\text{ratio}}}{N^q}.$$
Recall that for coefficients H technique, we need to compute a lower bound for

$$\Pr(X^q = x^q) \geq \frac{1 - \epsilon_{\text{ratio}}}{Nq}.$$ 

Mirror theory essentially provides the lower bound.

$$\Pr(\text{RP}_1(x_1) + \text{RP}_2(x_1) = y_1, \ldots, \text{RP}_1(x_q) + \text{RP}_2(x_q) = y_q)$$

$$\geq \frac{1 - O(q/N)}{Nq}.$$ 

Hence, $\text{Adv}^{\text{dist}}_{\mathcal{O}_1, \mathcal{O}_2}(\mathcal{A}) = O(q/N)$. 
What is Mirror theory?

1. Similar result with a single permutations.

2. The number of permutations $\pi$ such that $\pi(0|x_i) + \pi(1|x_i) = y_i$ is at least $\frac{N!^2}{N^q}$ for $q < N/67$.

   So $\epsilon_{\text{ratio}} = 0$. However, $y_i$’s are non-zero (need a bad set of transcripts and $\epsilon_{\text{bad}} = q/2^n$).

3. In other words, for all $q$-distinct $x^q$ and non-zero $y_i$’s,

$$\Pr(\text{RP}(0|x_1) + \text{RP}(1|x_1) = y_1, \ldots, \text{RP}(0|x_q) + \text{RP}(1|x_q) = y_q) \geq \frac{1}{N^q}.$$
Patarin considered the following general problem also called mirror theory.

1. distinct \( x_{i,j} \in \{0, 1\}^n, \ i \in [q], \ j \in [w] \) and

2. \( y_{i,j} \in \{0, 1\}^n. \ i \in [q], \ j \in [w] \) such that \( y_{i,j} \)'s are nonzero and for every \( i, \ y_{i,1}, \ldots, y_{i,w-1} \) are distinct.

\[
\Pr(\text{ for all } i, \ \text{RP}(x_{i,1}) \oplus \text{RP}(x_{i,w}) = y_{i,1}, \ldots, \\
\text{RP}(x_{i,w-1}) \oplus \text{RP}(x_{i,w}) = y_{i,w-1}) \geq \frac{1}{Nq}.
\]

This is also studied in CENC (by Tetsu Iwata, FSE 2006).
Key stream for CENC with $w = 2$, $w = 4$

CENC cipher with $w = 4$

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χ² Method

- X := (X₁, ..., Xₚ) and Y := (Y₁, ..., Yₚ) are two random vectors of size q distributed over Ωᵢ².

- \( P_{0|x_{i-1}}[x_i] = \Pr(X_i = x_i|X_1 = x_1, \ldots, X_{i-1} = x_{i-1}) \)

- \( P_{1|x_{i-1}}[x_i] = \Pr(Y_i = x_i|Y_1 = x_1, \ldots, Y_{i-1} = x_{i-1}) \)

- When \( i = 1 \), \( P_{0|x_{i-1}}[x_1] \) represents \( P[X_1 = x_1] \). Similarly, for \( P_{1|x_{i-1}}[x_1] \).
- Let $x^{i-1} \in \Omega^{i-1}, \ i \geq 1$.
- $\chi^2(\cdot)$ a real valued function defined as
  \[
  \chi^2(x^{i-1}) := d\chi^2(P_0|x^{i-1}, P_1|x^{i-1}).
  \]

- In other notation,
  \[
  \chi^2(x^{i-1}) := \sum_{x_i} \left( \frac{\Pr_X(x_i|x^{i-1}) - \Pr_Y(x_i|x^{i-1})}{\Pr_Y(x_i|x^{i-1})} \right)^2.
  \]
Let $x^{i-1} \in \Omega^{i-1}, i \geq 1$.

$\chi^2(\cdot)$ a real valued function defined as

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$$\chi^2(x^{i-1}) := \sum_{x_i} \frac{(\Pr_X(x_i|x^{i-1}) - \Pr_Y(x_i|x^{i-1}))^2}{\Pr_Y(x_i|x^{i-1})}.$$
Suppose $P_0$ and $P_1$ denote probability distributions of $X := (X_1, \ldots, X_q)$ and $Y := (Y_1, \ldots, Y_q)$ and for all $x_1, \ldots, x_{i-1}$, we have $P_0|_{x^{i-1}} \ll P_1|_{x^{i-1}}$. Then

$$\|P_0 - P_1\| \leq \left( \frac{1}{2} \sum_{i=1}^{q} \text{Ex}[\chi^2(X^i)] \right)^{\frac{1}{2}}.$$
Comparison with H-coefficient technique

1. Need: conditional probability instead of joint probabilities.

2. Suppose, for all \( x^q \) and \( i \leq q \),

\[
1 + \epsilon \geq \frac{\Pr_X(x_i|x^{i-1})}{\Pr_Y(x_i|x^{i-1})} \geq 1 - \epsilon
\]

3. Then, \( \frac{\Pr_X(x^q)}{\Pr_Y(x^q)} \geq 1 - q\epsilon \) and so \( \| \Pr_X - \Pr_Y \| \leq \epsilon \times q \).

4. If we apply \( \chi^2 \) method, \( \| \Pr_X - \Pr_Y \| \leq \epsilon \times \sqrt{q/2} \).

5. If we know more on the distributions get better bound.
Switching between PRF and PRP

1. \( \Pr_Y(x_i | x^{i-1}) = \frac{1}{2^n} \) for all \( i \)-distinct \( x^i \)

   \[
   \Pr_X(x_i | x^{i-1}) = \frac{1}{(2^n - i + 1)} \quad \text{if } x_i \not\in x^{i-1}
   \]
   \[
   = 0 \quad \text{if } x_i \in x^{i-1}
   \]

2. \[
   \left( \frac{\Pr_X(x_i | x^{i-1}) - \Pr_Y(x_i | x^{i-1})}{\Pr_Y(x_i | x^{i-1})} \right)^2 = \frac{(i - 1)^2}{2^n(2^n - i + 1)^2} \]
   \[
   = \frac{1}{2^n} \quad \text{if } x_i \not\in x^{i-1}
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2. \[
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\]
\[
= \frac{1}{2^n} \quad \text{if } x_i \in x^{i-1}
\]
Switching between PRF and PRP

\[ \chi^2(x^{i-1}) = \sum_{x_i} \left( \frac{\Pr_X(x_i|x^{i-1}) - \Pr_Y(x_i|x^{i-1})}{\Pr_Y(x_i|x^{i-1})} \right)^2 \]

\[ = \frac{i - 1}{2^n} + \frac{(i - 1)^2}{2^n(2^n - i + 1)}. \]

By \( \chi^2 \) method,

\[ \| \Pr_X - \Pr_Y \| \leq \sum_{i=1}^{q} \frac{1}{2} (\mathbb{E}_X(\chi^2(x^{i-1})))^{1/2} \]

\[ = \sqrt{\frac{q(q - 1)}{2^{n+1}}} + \frac{q^3}{2^{2n}}. \]
Probability in Cryptography

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### Comparisons

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<th>using mirror Th.</th>
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<td>$(q^3/2^{2n})^{1/2}$</td>
<td>$q/2^n$</td>
<td>$(q^4/2^{3n})^{1/2}$</td>
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<tr>
<td>XORP</td>
<td>-</td>
<td>$q/2^n$</td>
<td>$q/2^n$</td>
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<td>-</td>
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<td>$(q/2^n - m/2)^{2/3}$</td>
<td>-</td>
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Outline of the talk

1 Probability in Cryptography
   - Well Known Distribution in Cryptography
   - Some Metrics on Probability Distributions

2 Two Tools: H-Coefficient and $\chi^2$
   - H-Coefficient Technique
   - Mirror theory
   - $\chi^2$ Method

3 Some Constructions and Applications
   - Encrypted Davies-Meyer (EDM) Construction
   - Truncation Construction
   - Sum of Permutations Construction
Encrypted Davies-Meyer (EDM) Construction

\[ \text{EDM}_{\pi,\pi'} : \{0, 1\}^n \times \{0, 1\}^n \mapsto \{0, 1\}^n \]

- Takes two permutations \( \pi, \pi' \in \text{Perm}_n \) as key.
- On input \( x \in \{0, 1\}^n \), returns \( \pi'(\pi(x) \oplus x) \).

Bound using coefficients \( H \) technique (Cogliati and Seurin - Crypto 2016)

\[ \text{Adv}^\text{prf}_{\text{EDM}}(A) \leq \frac{5q^3}{2N}. \]

Bound using \( \chi^2 \) method (Dai, Hoang, Tessaro - Crypto 2017)

\[ \text{Adv}^\text{prf}_{\text{EDM}}(A) \leq \frac{3q^2}{N^{3/2}}. \]
Proof Sketch: $\text{EDM}_{\pi, \pi'}(x) = \pi'(\pi(x) \oplus x)$

**Upper bound**

$\Pr_X(x_i | x^{i-1}) \leq 1/(2^n - i) \leq \frac{1}{2^n} + \frac{2i}{2^{2n}}$. 

**Lower bound**

$\Pr_X(x_i | x^{i-1}) \geq \frac{2^n - 4i}{2^n(2^n - i)} \geq \frac{1}{2^n} - \frac{4i}{2^{2n}}$. 

$|\Pr_X(x_i | x^{i-1}) - \frac{1}{2^n}| \leq \frac{4i}{2^{2n}}$. 

- $\chi^2(X^{i-1}) \leq \frac{16i^3}{2^{3n}}$ (non-random bound).
- $\sum_i \text{Ex}(\chi^2(X^{i-1})) \leq \frac{18q^4}{2^{3n}}$. So, $\text{Adv}_{\text{EDM}}^{\text{prf}}(\mathcal{A}) \leq \frac{3q^2}{N^2}$. 


Proof Sketch: \( \text{EDM}_{\pi, \pi'}(x) = \pi'(\pi(x) \oplus x) \)

**Upper bound** \( \Pr_X(x_i | x^{i-1}) \leq 1/(2^n - i) \leq \frac{1}{2^n} + \frac{2i}{2^{2n}}. \)

**Lower bound** \( \Pr_X(x_i | x^{i-1}) \geq \frac{2^n - 4i}{2^n (2^n - i)} \geq \frac{1}{2^n} - \frac{4i}{2^{2n}}. \)

\[
\left| \Pr_X(x_i | x^{i-1}) - \frac{1}{2^n} \right| \leq \frac{4i}{2^{2n}}. 
\]

- \( \chi^2(X^{i-1}) \leq \frac{16i^3}{2^{3n}} \) (non-random bound).

- \( \sum_i \mathbb{E}_x(\chi^2(X^{i-1})) \leq \frac{18q^4}{2^{3n}}. \) So, \( \text{Adv}^{\text{prf}}_{\text{EDM}}(A) \leq \frac{3q^2}{N^2}. \)
Proof Sketch: $\text{EDM}_{\pi, \pi'}(x) = \pi'(\pi(x) \oplus x)$

**Upper bound**\n\[ \Pr_X(x_i | x^{i-1}) \leq \frac{1}{2^n} + \frac{2i}{2^{2n}}. \]

**Lower bound**\n\[ \Pr_X(x_i | x^{i-1}) \geq \frac{2^n - 4i}{2^n(2^n - i)} \geq \frac{1}{2^n} - \frac{4i}{2^{2n}}. \]

\[ |\Pr_X(x_i | x^{i-1}) - \frac{1}{2^n}| \leq \frac{4i}{2^{2n}}. \]

\[ \chi^2(X^{i-1}) \leq \frac{16i^3}{2^{3n}} \text{ (non-random bound)}. \]

\[ \sum_i \text{Ex}(\chi^2(X^{i-1})) \leq \frac{18q^4}{2^{3n}}. \text{ So, } \text{Adv}^\text{prf}_{\text{EDM}}(A) \leq \frac{3q^2}{N^{\frac{3}{2}}}. \]
Proof Sketch: \( \text{EDM}_{\pi, \pi'}(x) = \pi'(\pi(x) \oplus x) \)

- **upper bd** \( \Pr_X(x_i | x^{i-1}) \leq 1/(2^n - i) \leq \frac{1}{2^n} + \frac{2i}{2^{2n}}. \)

- **lower bd** \( \Pr_X(x_i | x^{i-1}) \geq \frac{2^n - 4i}{2^n (2^n - i)} \geq \frac{1}{2^n} - \frac{4i}{2^{2n}}. \)

\[
| \Pr_X(x_i | x^{i-1}) - \frac{1}{2^n} | \leq \frac{4i}{2^{2n}}.
\]

- \( \chi^2(X^{i-1}) \leq \frac{16i^3}{2^{3n}} \) (non-random bound).

- \( \sum_i \mathbf{Ex}(\chi^2(X^{i-1})) \leq \frac{18q^4}{2^{3n}}. \) So, \( \text{Adv}^{\text{prf}}_{\text{EDM}}(A) \leq \frac{3q^2}{N^{3/2}}. \)
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3. Some Constructions and Applications
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Construction

1. Let $m \leq n$ and $\text{trunc}_m$ denote the function which returns the first $m$ bits of $x \in \{0, 1\}^n$.

2. We define for every $x \in \{0, 1\}^n$, 
   \[
   \text{trRP}_m(x) = \text{trunc}_m(\text{RP}_n(x)).
   \]

Note that it is a function family, keyed by random permutation, mapping the set of all $n$ bits to the set of all $m$ bits.

3. Let $X_1, \ldots, X_q$ denote all outputs of the construction to the adversary then $X_i = \text{trunc}_m(V_i)$ for all $i$. 
Proof Sketch: \( \text{trRP}_m(x) = \text{trunc}_m(\text{RP}(x)) \)

- \( \Pr_X(x_i|x^{i-1}) = \frac{2^n-m-H}{2^n-i+1} \) where \( H \) follows Hypergeometric distribution (HG).

- \( \chi^2(x^{i-1}) = \sum x \frac{2^m}{(2^n-i+1)^2} \times \left( H - \frac{i-1}{2^m} \right)^2 \)

- By using expectation and variance formula of HG and \( \chi^2 \) method, we have

\[
\text{Adv}^{\text{prf}}_{\text{trRP}_m}(\mathcal{A}) \leq \left( \frac{1}{2} \sum_{i=1}^{q} \mathbb{E}[\chi^2(X^{i-1})] \right)^{\frac{1}{2}} \leq \frac{q \times 2^{(m-1)/2}}{2^n}.
\]
Theorem for $\text{trRP}_m$

**Theorem**

*For any adversary $A$ making $q$ queries we have*

$$\text{Adv}^{\text{prf}}_{\text{trRP}_m}(A) \leq \frac{q \times 2^{(m-1)/2}}{2^n}.$$ 

1. When, $m = n$ (no truncation), PRF advantage is $O(q/2^{n/2})$ (again, the presence of square root).

2. When $m = 1$ (returns only one bit), PRF advantage is $O(q/2^n)$.

3. When $m = n/2$ (mid-way: returns half of the bits), PRF advantage is $O(q/2^{3n/4})$. 

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XOR Construction

1 Define $\text{XOR}_\pi : \{0, 1\}^{n-1} \rightarrow \{0, 1\}^n$ to be the construction that takes a permutation $\pi \in \text{Perm}_n$ as a key, and on input $x \in \{0, 1\}^{n-1}$ it returns $\pi(x\|0) \oplus \pi(x\|1)$.

2 XOR construction based on a random permutation $\text{RP}_n$ returns $X_1, \ldots, X_q$ where $X_1 := V_1 \oplus V_2$, $X_2 := V_2 \oplus V_3$, $\ldots$, $X_q := V_{2q-1} \oplus V_{2q}$ and $V_1, \ldots, V_{2q} \leftarrow \text{wor} \{0, 1\}^n$.

3 Mirror theory and H-coefficients proves the PRF security.
Sum of Permutations.

Theorem (DHT-Crypto-17)

Fix an integer $n \geq 8$ and let $N = 2^n$. For any adversary $A$ that makes $q \leq \frac{N}{32}$ queries we have

$$\text{Adv}_{\text{XOR}}^{\text{prf}}(A) \leq \frac{1.5q + 3\sqrt{q}}{N}.$$ 

1. $U_1', \ldots, U_q' \leftarrow \{0, 1\}^n$.

2. Let $P_1$ and $P_2$ denote the output distributions of $X := (X_1, \ldots, X_q)$ and $U' := (U_1', \ldots, U_q')$ respectively. Thus,

$$\text{Adv}_{\text{XOR}}^{\text{prf}}(A) \leq \|P_1 - P_2\|.$$
Sum of Permutations.

1. $P_0$ is the probability function for 
   $(U_1, \ldots, U_q) \leftarrow \text{wr} \left[ N \right]^* := \{0, 1\}^n \setminus \{0^n\}$.

2. $\|P_0 - P_2\| \leq q/2^n$.

3. It is sufficient to bound $\|P_0 - P_1\|$.

4. For every non-zero $x_1, \ldots, x_i$ we clearly have 
   $P_0|_{x^{i-1}}(x_i) = 1/(N - 1)$.

   \[
   \chi^2(x^{i-1}) = \sum_{x \neq 0^n} (N - 1)(Y_{i,x} - \frac{1}{N - 1})^2. \tag{1}
   \]

   where $Y_{i,x} := \text{Pr}(X_i = x | X^{i-1} = x^{i-1})$. 
Sum of Permutations.

1. \( S = \{V_1, V_2, \ldots, V_{2i-2}\} \).

2. Let \( D_{i,x} \) be the number of pairs \((u, u \oplus x)\) such that both \( u \) and \( u \oplus x \) belongs to \( S \).

3. Note that \( S \) and \( D_{i,x} \) are both random variables, and in fact functions of the random variables \( V_1, V_2, \ldots, V_{2i-2} \).

\[
\gamma_{i,x} = \frac{N - 4(i - 1) + D_{i,x}}{(N - 2i + 1)(N - 2i)}. \quad (2)
\]
Sum of Permutations.

1

\[
(Y_{i,x} - \frac{1}{N-1})^2 \leq \frac{3(D_{i,x} - 4(i-1)^2/N)^2 + 18}{N^4}.
\]

\[
E_x(\chi^2(X^{i-1})) \leq \sum_{x \neq 0^n} N \cdot E_x[(Y_{i,x} - \frac{1}{N-1})^2]
\]

\[
\leq \sum_{x \neq 0^n} \frac{18}{N^3} + \frac{3}{N^3} \cdot E_x[(D_{i,x} - \frac{4(i-1)^2}{N})^2]
\]

2

\(D_{i,x}\) as a function of \(V_1, V_2, \ldots, V_{2i-2}\), and the expectation is taken over the choices of \(V_1, V_2, \ldots, V_{2i-2}\).
\[
\text{Ex}[(D_{i,x} - \frac{4(i-1)^2}{N})^2] \leq \frac{4(i-1)^2}{N} \tag{5}
\]

\[
\text{Ex}(\chi^2(X^{i-1})) \leq \frac{18}{N^2} + \frac{12(i-1)^2}{N^3}.
\]

Summing up, from \(\chi^2\)-method

\[
\|P_0 - P_1\| \leq \left( \frac{1}{2} \sum_{i=1}^{q} \text{Ex}[\chi^2(X^{i-1})] \right)^{\frac{1}{2}}
\leq 3\sqrt{q} + .5q \leq \frac{3\sqrt{q} + .5q}{N}.
\]
1. Is everything OK?

2. We have

\[ P[X_i = x | V_1 = v_1, \ldots, V_{2i-2} = v_{2i-2}] = \frac{N - 4(i - 1) + D_{i,x}}{(N - 2i + 1)(N - 2i)} \]  

(6)

But,

\[ P[X_i = x | V_{2i-2} = v_{2i-2}] = P[X_i = x | X^{i-1} = x^{i-1}] \]  

(7)

does not hold for every \( v_1, \ldots, v_{2i-2} \).
1. Is everything OK?

2. we have

\[ P[X_i = x | V_1 = v_1, \ldots, V_{2i-2} = v_{2i-2}] = \frac{N - 4(i - 1) + D_{i,x}}{(N - 2i + 1)(N - 2i)} \] (6)

But,

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does not hold for every \( v_1, \ldots, v_{2i-2} \).
How to get rid of it?

1. Consider an extended system which leaks more (similar to H technique).

2. Release $V_i$ values in real world. In the ideal world simulate the $V_i$ values keeping compatibility.

3. We aim a more general useful form of Mirror theory.
Summing Up

1. H-Technique is nowadays in popular (in comparison with game playing technique).

2. Sometimes hard to get optimum bound.

3. $\chi^2$ method can be another useful tool for proving security - mainly for close to optimal security.

4. Mirror theory needs attention. It has high potential,

5. We should also study the potentiality of the other metrics.
\[ h''_{\alpha+2} = h_\alpha + (-4a + 8) [h'_\alpha] u_1 \text{ (i.e. first blue term)} + [2\delta(\mu_1) + 2\delta(\mu_2) + 2\delta(\mu_3) + 2\delta(\mu_4) + 2\delta(\mu_1 \oplus \theta) + 2\delta(\mu_2 \oplus \theta) + 2\delta(\mu_3 \oplus \theta) + 2\delta(\mu_4 \oplus \theta)] [h'_\alpha] \]

(\text{i.e. terms with a value } \lambda(i) \text{ not compatible with } \varphi = 1 \text{ equation})

\[ + [2\delta(\mu_1 \oplus \mu_2) + 2\delta(\mu_1 \oplus \mu_3) + 2\delta(\mu_1 \oplus \mu_4) + 2\delta(\mu_2 \oplus \mu_3) + 2\delta(\mu_2 \oplus \mu_4) + 2\delta(\mu_3 \oplus \mu_4)] [h'_\alpha] \text{ (i.e. first green terms)} \]

\[ + 6(a - 2)(a - 4) [h''_\alpha] u_2 \text{ (i.e. blue term with } \varphi = 2 \text{ equations)} \]

\[- 15 \cdot 2 \cdot 3 \cdot (2\Delta)a [h'_\alpha] u_3 \text{ ("first red term", i.e. with } \varphi = 2)\]

\[ + 4\Delta u_4 [h'_\alpha] \text{ (i.e. green term: one dependent equation with } \varphi = 2) \]

\[ - 8\Delta u_5 [h''_\alpha] \text{ (i.e. green term one dependent equation with } \varphi = 3) \]

\[ - 4(a - 2)(a - 4)(a - 6)u_6 [h^{(3)}_\alpha] \text{ (i.e. blue term with } \varphi = 3) \]

\[ + 256a^2 \Delta u_7 [h^{(3)}_\alpha] \text{ (i.e. red term with } \varphi = 3) \]

\[ + (a - 2)(a - 4)(a - 6)(a - 8)u_8 [h^{(4)}_\alpha] \text{ (i.e. blue term with } \varphi = 4) \]

\[- 90a^3 \Delta u_9 [h^{(4)}_\alpha] u_9 \text{ (i.e. red term with } \varphi = 4) \]

\[ + 12a^2 \Delta u_{10} [h^{(3)}_\alpha] \text{ (i.e. green term: one dependent equation with } \varphi = 4) \]

\[ + 36 \cdot (2\Delta)^2 u_{11} [h''_\alpha] \text{ (i.e. green term: two dependent equations with } \varphi = 4) \]